# 2

# Rigid-Body Kinematics




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## 2.1 Rotations in Three Dimensions

Spatial rigid-body rotations are defined as motions that preserve the distance between points in a body before and after the motion and leave one point fixed under the motion. By definition a motion must be physically realizable, and so reflections are not allowed. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are any two points in a body before a rigid motion, then $\mathbf{x}_1$, and $\mathbf{x}_2$ are the corresponding points after rotation, and

$$d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{X}_1, \mathbf{X}_2)$$

where

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

is the *Euclidean* distance. We view the transformation from $\mathbf{X}_i$ to $\mathbf{x}_i$ as a function $\mathbf{x}(\mathbf{X}, t)$.

By appropriately choosing our frame of reference in space, it is possible to make the pivot point (the point which does not move under rotation) the origin. Therefore, $\mathbf{x}(\mathbf{0}, t) = \mathbf{0}$. With this choice, it can be shown that a necessary condition for a motion to be a rotation is

$$\mathbf{x}(\mathbf{X}, t) = A(t)\mathbf{X}$$

where $A(t) \in I\!R^{3\times 3}$ is a time-dependent matrix.

Constraints on the form of $A(t)$ arise from the distance-preserving properties of rotations. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are vectors defined in the frame of reference attached to the pivot, then the triangle with sides of length


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$||\mathbf{X}_1||$, $||\mathbf{X}_2||$, and $||\mathbf{X}_1 - \mathbf{X}_2||$ is congruent to the triangle with sides of length $||\mathbf{x}_1||$, $||\mathbf{x}_2||$, and $||\mathbf{x}_1 - \mathbf{x}_2||$. Hence, the angle between the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ must be the same as the angle between $\mathbf{X}_1$ and $\mathbf{X}_2$. In general $\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}|| \, ||\mathbf{y}|| \cos\theta$ where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Since $||\mathbf{x}_i|| = ||\mathbf{X}_i||$ in our case, it follows that

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{X}_1 \cdot \mathbf{X}_2$$

Observing that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$ and $\mathbf{x}_i = A\mathbf{X}_i$, we see that

$$(A\mathbf{X}_1)^T (A\mathbf{X}_2) = \mathbf{X}_1^T \mathbf{X}_2 \tag{2.1}$$

Moving everything to the left side of the equation, and using the transpose rule for matrix vector multiplication, Equation (2.1) is rewritten as

$$\mathbf{X}_1^T (A^T A - \mathbb{I})\mathbf{X}_2 = 0$$

where $\mathbb{I}$ is the $3 \times 3$ identity matrix.

Since $\mathbf{X}_1$ and $\mathbf{X}_2$ were arbitrary points to begin with, this holds *for all* possible choices. The only way this can hold is if

$$A^T A = \mathbb{I} \tag{2.2}$$

An easy way to see this is to choose $\mathbf{X}_1 = \mathbf{e}_i$ and $\mathbf{X}_2 = \mathbf{e}_j$ for $i, j \in \{1, 2, 3\}$. This forces all the components of the matrix $A^T A - \mathbb{I}$ to be zero.

Equation (2.2) says that a rotation matrix is one whose inverse is its transpose. Taking the determinant of both sides of this equation yields $(\det A)^2 = 1$. There are two possibilities: $\det A = \pm 1$. The case $\det A = -1$ is a reflection and is not physically realizable in the sense that a rigid body cannot be reflected (only its image can be). A rotation is what remains:

$$\det A = +1 \tag{2.3}$$

Thus, a rotation matrix $A$ is one which satisfies both Equation (2.2) and Equation (2.3). The set of all real matrices satisfying both Equation (2.2) and Equation (2.3) is called the set of *special orthogonal*[1] matrices. In general, the set of all $N \times N$ special orthogonal matrices is called $SO(N)$, and the set of all rotations in three-dimensional space is referred to as $SO(3)$.

In the special case of rotation about a fixed axis by an angle $\phi$, the rotation has only one degree of freedom. In particular, for counterclockwise rotations about the $\mathbf{e}_3$, $\mathbf{e}_2$, and $\mathbf{e}_1$ axes:

$$R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4}$$

$$R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \tag{2.5}$$

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{2.6}$$

[1] Also called proper orthogonal.

### 2.1.1 Rules for Composing Rotations

Consider three frames of reference $A$, $B$, and $C$, all of which have the same origin. The vectors $\mathbf{x}^A, \mathbf{x}^B, \mathbf{x}^C$ represent *the same* arbitrary point in space, $\mathbf{x}$, as it is viewed in the three different frames. With respect to some common frame fixed in space with axes defined by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $(\mathbf{e}_i)_j = \delta_{ij}$, the rotation matrices describing the basis vectors of the frames $A$, $B$, and $C$ are

$$R_A = \left[\mathbf{e}_1^A, \mathbf{e}_2^A, \mathbf{e}_3^A\right] \quad R_B = \left[\mathbf{e}_1^B, \mathbf{e}_2^B, \mathbf{e}_3^B\right] \quad R_C = \left[\mathbf{e}_1^C, \mathbf{e}_2^C, \mathbf{e}_3^C\right]$$

where the vectors $\mathbf{e}_i^A$, $\mathbf{e}_i^B$, and $\mathbf{e}_i^C$ are unit vectors along the $i^{\text{th}}$ axis of frame $A$, $B$, or $C$. The "absolute" coordinates of the vector $\mathbf{x}$ are then given by

$$\mathbf{x} = R_A\mathbf{x}^A = R_B\mathbf{x}^B = R_C\mathbf{x}^C$$

In this notation, which is often used in the field of robotics (see e.g., [1, 2]), there is effectively a "cancellation" of indices along the upper right to lower left diagonal.

Given the rotation matrices $R_A$, $R_B$, and $R_C$, it is possible to define rotations of one frame *relative* to another by observing that, for instance, $R_A\mathbf{x}^A = R_B\mathbf{x}^B$ implies $\mathbf{x}^A = (R_A)^{-1}R_B\mathbf{x}^B$. Therefore, given any vector $\mathbf{x}^B$ as it looks in $B$, we can find how it looks in $A$, $\mathbf{x}^A$, by performing the transformation:

$$\mathbf{x}^A = R_B^A\mathbf{x}^B \quad \text{where} \quad R_B^A = (R_A)^{-1}R_B \tag{2.7}$$

It follows from substituting the analogous expression $\mathbf{x}^B = R_C^B\mathbf{x}^C$ into $\mathbf{x}^A = R_B^A\mathbf{x}^B$ that concatenation of rotations is calculated as

$$\mathbf{x}^A = R_C^A\mathbf{x}^C \quad \text{where} \quad R_C^A = R_B^A R_C^B \tag{2.8}$$

Again there is effectively a cancellation of indices, and this propagates through for any number of relative rotations. Note that the order of multiplication is critical.

In addition to changes of basis, rotation matrices can be viewed as descriptions of motion. Multiplication of a rotation matrix $Q$ (which represents a frame of reference) by a rotation matrix $R$ (representing motion) on the left, $RQ$, has the effect of moving $Q$ by $R$ relative to the *base frame*. Multiplying by the same rotation matrix on the right, $QR$, has the effect of moving by $R$ relative to the the frame $Q$.

To demonstrate the difference, consider a frame of reference $Q = [\mathbf{a}, \mathbf{b}, \mathbf{n}]$ where $\mathbf{a}$ and $\mathbf{b}$ are unit vectors orthogonal to each other, and $\mathbf{a} \times \mathbf{b} = \mathbf{n}$. First rotating from the identity $\mathbb{I} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ fixed in space to $Q$ and then rotating relative to $Q$ by $R_3(\theta)$ results in $QR_3(\theta)$. On the other hand, a rotation about the vector $\mathbf{e}_3^Q = \mathbf{n}$ as viewed in the fixed frame is a rotation $A(\theta, \mathbf{n})$. Hence, shifting the frame of reference $Q$ by multiplying on the left by $A(\theta, \mathbf{n})$ has the same effect as $QR_3(\theta)$, and so we write

$$A(\theta, \mathbf{n})Q = QR_3(\theta) \quad \text{or} \quad A(\theta, \mathbf{n}) = QR_3(\theta)Q^T \tag{2.9}$$

This is one way to define the matrix

$$A(\theta, \mathbf{n}) = \begin{pmatrix} n_1^2 v\theta + c\theta & n_2 n_1 v\theta - n_3 s\theta & n_3 n_1 v\theta + n_2 s\theta \\ n_1 n_2 v\theta + n_3 s\theta & n_2^2 v\theta + c\theta & n_3 n_2 v\theta - n_1 s\theta \\ n_1 n_3 v\theta - n_2 s\theta & n_2 n_3 v\theta + n_1 s\theta & n_3^2 v\theta + c\theta \end{pmatrix}$$

where $s\theta = \sin\theta$, $c\theta = \cos\theta$, and $v\theta = 1 - \cos\theta$. This expresses a rotation in terms of its axis and angle, and is a mathematical statement of *Euler's Theorem*.

Note that $\mathbf{a}$ and $\mathbf{b}$ do not appear in the final expression. There is nothing magical about $\mathbf{e}_3$, and we could have used the same construction using any other basis vector, $\mathbf{e}_i$, and we would get the same result so long as $\mathbf{n}$ is in the $i$th column of $Q$.

### 2.1.2 Euler Angles

Euler angles are by far the most widely known parametrization of rotation. They are generated by three successive rotations about independent axes. Three of the most common choices are the $ZXZ$, $ZYZ$, and $ZYX$ Euler angles. We will denote these as

$$A_{ZXZ}(\alpha, \beta, \gamma) = R_3(\alpha)R_1(\beta)R_3(\gamma) \tag{2.10}$$
$$A_{ZYZ}(\alpha, \beta, \gamma) = R_3(\alpha)R_2(\beta)R_3(\gamma) \tag{2.11}$$
$$A_{ZYX}(\alpha, \beta, \gamma) = R_3(\alpha)R_2(\beta)R_1(\gamma) \tag{2.12}$$

Of these, the $ZXZ$ and $ZYZ$ Euler angles are the most common, and the corresponding matrices are explicitly

$$A_{ZXZ}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\gamma\cos\alpha - \sin\gamma\sin\alpha\cos\beta & -\sin\gamma\cos\alpha - \cos\gamma\sin\alpha\cos\beta & \sin\beta\sin\alpha \\ \cos\gamma\sin\alpha + \sin\gamma\cos\alpha\cos\beta & -\sin\gamma\sin\alpha + \cos\gamma\cos\alpha\cos\beta & -\sin\beta\cos\alpha \\ \sin\beta\sin\gamma & \sin\beta\cos\gamma & \cos\beta \end{pmatrix}$$

and

$$A_{ZYZ}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\gamma\cos\alpha\cos\beta - \sin\gamma\sin\alpha & -\sin\gamma\cos\alpha\cos\beta - \cos\gamma\sin\alpha & \sin\beta\cos\alpha \\ \sin\alpha\cos\gamma\cos\beta + \sin\gamma\cos\alpha & -\sin\gamma\sin\alpha\cos\beta + \cos\gamma\cos\alpha & \sin\beta\sin\alpha \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}$$

The ranges of angles for these choices are $0 \le \alpha \le 2\pi$, $0 \le \beta \le \pi$, and $0 \le \gamma \le 2\pi$. When $ZYZ$ Euler angles are used,

$$\begin{aligned} R_3(\alpha)R_2(\beta)R_3(\gamma) &= R_3(\alpha)(R_3(\pi/2)R_1(\beta)R_3(-\pi/2))R_3(\gamma) \\ &= R_3(\alpha + \pi/2)R_1(\beta)R_3(-\pi/2 + \gamma) \end{aligned}$$

and so

$$R_{ZYZ}(\alpha, \beta, \gamma) = R_{ZXZ}(\alpha + \pi/2, \beta, \gamma - \pi/2)$$

### 2.1.3 The Matrix Exponential

The result of Euler's theorem discussed earlier can be viewed in another way using the concept of a matrix exponential. Recall that the Taylor series expansion of the scalar exponential function is

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

The matrix exponential is the same formula evaluated at a square matrix:

$$e^X = \mathbb{I} + \sum_{k=1}^{\infty} \frac{X^k}{k!}$$

Let $N\mathbf{y} = \mathbf{n} \times \mathbf{y}$ for the unit vector $\mathbf{n}$ and any $\mathbf{y} \in I\!R^3$, and let this relationship be denoted as $\mathbf{n} = \mathrm{vect}(N)$. It may be shown that $N^2 = \mathbf{n}\mathbf{n}^T - \mathbb{I}$.

All higher powers of $N$ can be related to either $N$ or $N^2$ as

$$N^{2k+1} = (-1)^k N \quad \text{and} \quad N^{2k} = (-1)^{k+1} N^2 \tag{2.13}$$

The first few terms in the Taylor series of $e^{\theta N}$ are then expressed as

$$e^{\theta N} = \mathbb{I} + (\theta - \theta^3/3! + \cdots)N + (\theta^2/2! - \theta^4/4! + \cdots)N^2$$

Hence for any rotational displacement, we can write

$$A(\theta, \mathbf{n}) = e^{\theta N} = \mathbb{I} + \sin\theta N + (1 - \cos\theta) N^2$$

This form clearly illustrates that $(\theta, \mathbf{n})$ and $(-\theta, -\mathbf{n})$ correspond to the same rotation.

Since $\theta = \|\mathbf{x}\|$ where $\mathbf{x} = \text{vect}(X)$ and $N = X/\|\mathbf{x}\|$, one sometimes writes the alternative form

$$e^X = \mathbb{I} + \frac{\sin\|\mathbf{x}\|}{\|\mathbf{x}\|} X + \frac{(1 - \cos\|\mathbf{x}\|)}{\|\mathbf{x}\|^2} X^2$$

## 2.2 Full Rigid-Body Motion

The following statements address what comprises the complete set of rigid-body motions.

> Chasles' Theorem [12]: (1) Every motion of a rigid body can be considered as a translation in space and a rotation about a point; (2) Every spatial displacement of a rigid body can be equivalently affected by a single rotation about an axis and translation along the same axis.

In modern notation, (1) is expressed by saying that every point $\mathbf{x}$ in a rigid body may be moved as

$$\mathbf{x}' = R\mathbf{x} + \mathbf{b} \tag{2.14}$$

where $R \in SO(3)$ is a rotation matrix, and $\mathbf{b} \in I\!R^3$ is a translation vector.

The pair $g = (R, \mathbf{b}) \in SO(3) \times I\!R^3$ describes both motion of a rigid body *and* the relationship between frames fixed in space and in the body. Furthermore, motions characterized by a pair $(R, \mathbf{b})$ could describe the behavior either of a rigid body or of a deformable object undergoing a rigid-body motion during the time interval for which this description is valid.

### 2.2.1 Composition of Motions

Consider a rigid-body motion which moves a frame originally coincident with the "natural" frame $(\mathbb{I}, \mathbf{0})$ to $(R_1, \mathbf{b}_1)$. Now consider a relative motion of the frame $(R_2, \mathbf{b}_2)$ with respect to the frame $(R_1, \mathbf{b}_1)$. That is, given any vector $\mathbf{x}$ defined in the terminal frame, it will look like $\mathbf{x}' = R_2\mathbf{x} + \mathbf{b}_2$ in the frame $(R_1, \mathbf{b}_1)$. Then the same vector will appear in the natural frame as

$$\mathbf{x}'' = R_1(R_2\mathbf{x} + \mathbf{b}_2) + \mathbf{b}_1 = R_1 R_2 \mathbf{x} + R_1\mathbf{b}_2 + \mathbf{b}_1$$

The net effect of composing the two motions (or changes of reference frame) is equivalent to the definition

$$(R_3, \mathbf{b}_3) = (R_1, \mathbf{b}_1) \circ (R_2, \mathbf{b}_2) \triangleq (R_1 R_2, R_1\mathbf{b}_2 + \mathbf{b}_1) \tag{2.15}$$

From this expression, we can calculate the motion $(R_2, \mathbf{b}_2)$ that for any $(R_1, \mathbf{b}_1)$ will return the floating frame to the natural frame. All that is required is to solve $R_1 R_2 = \mathbb{I}$ and $R_1\mathbf{b}_2 + \mathbf{b}_1 = \mathbf{0}$ for the variables $R_2$ and $\mathbf{b}_2$, given $R_1$ and $\mathbf{b}_1$. The result is $R_2 = R_1^T$ and $\mathbf{b}_2 = -R_1^T\mathbf{b}_1$. Thus, we denote the inverse of a motion as

$$(R, \mathbf{b})^{-1} = (R^T, -R^T\mathbf{b}) \tag{2.16}$$

This inverse, when composed either on the left or the right side of $(R, \mathbf{b})$, yields $(\mathbb{I}, \mathbf{0})$.

The set of all pairs $(R, \mathbf{b})$ together with the operation $\circ$ is denoted as $SE(3)$ for "Special Euclidean" group.

Note that every rigid-body motion (element of $SE(3)$) can be decomposed into a pure translation followed by a pure rotation as

$$(R, \mathbf{b}) = (\mathbb{I}, \mathbf{b}) \circ (R, \mathbf{0})$$

and every translation *conjugated*[2] by a rotation yields a translation:

$$(R, \mathbf{0}) \circ (\mathbb{I}, \mathbf{b}) \circ (R^T, \mathbf{0}) = (\mathbb{I}, R\mathbf{b}) \tag{2.17}$$

### 2.2.2 Screw Motions

The axis in the second part of Chasles' theorem is called the *screw axis*. It is a line in space about which a rotation is performed and along which a translation is performed.[3] As with any line in space, its direction is specified completely by a unit vector, $\mathbf{n}$, and the position of any point $\mathbf{p}$ on the line. Hence, a line is parametrized as

$$\mathbf{L}(t) = \mathbf{p} + t\mathbf{n}, \quad \forall t \in \mathbb{R}$$

Since there are an infinite number of vectors $\mathbf{p}$ on the line that can be chosen, the one which is "most natural" is that which has the smallest magnitude. This is the vector originating at the origin of the coordinate system and terminating at the line which it intersects orthogonally.

Hence the condition $\mathbf{p} \cdot \mathbf{n} = 0$ is satisfied. Since $\mathbf{n}$ is a unit vector and $\mathbf{p}$ satisfies a constraint equation, a line is uniquely specified by only four parameters. Often instead of the pair of line coordinates $(\mathbf{n}, \mathbf{p})$, the pair $(\mathbf{n}, \mathbf{p} \times \mathbf{n})$ is used to describe a line because this implicitly incorporates the constraint $\mathbf{p} \cdot \mathbf{n} = 0$. That is, when $\mathbf{p} \cdot \mathbf{n} = 0$, $\mathbf{p}$ can be reconstructed as $\mathbf{p} = \mathbf{n} \times (\mathbf{p} \times \mathbf{n})$, and it is clear that for unit $\mathbf{n}$, the pair $(\mathbf{n}, \mathbf{p} \times \mathbf{n})$ has four degrees of freedom. Such a description of lines is called the Plücker coordinates.

Given an arbitrary point $\mathbf{x}$ in a rigid body, the transformed position of the same point after translation by $d$ units along a screw axis with direction specified by $\mathbf{n}$ is $\mathbf{x}' = \mathbf{x} + d\mathbf{n}$. Rotation about the same screw axis is given as $\mathbf{x}'' = \mathbf{p} + e^{\theta N}(\mathbf{x}' - \mathbf{p})$.

Since $e^{\theta N}\mathbf{n} = \mathbf{n}$, it does not matter if translation along a screw axis is performed before or after rotation. Either way, $\mathbf{x}'' = \mathbf{p} + e^{\theta N}(\mathbf{x} - \mathbf{p}) + d\mathbf{n}$.

It may be shown that the screw axis parameters $(\mathbf{n}, \mathbf{p})$ and motion parameters $(\theta, d)$ always can be extracted from a given rigid displacement $(R, \mathbf{b})$.

## 2.3 Homogeneous Transforms and the Denavit-Hartenberg Parameters

It is of great convenience in many fields, including robotics, to represent each rigid-body motion with a transformation matrix instead of a pair of the form $(R, \mathbf{b})$ and to use matrix multiplication in place of a composition rule.

### 2.3.1 Homogeneous Transformation Matrices

One can assign to each pair $(R, \mathbf{b})$ a unique $4 \times 4$ matrix

$$H(R, \mathbf{b}) = \begin{pmatrix} R & \mathbf{b} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{2.18}$$

This is called a homogeneous transformation matrix, or simply a *homogeneous transform*. It is easy to see by the rules of matrix multiplication and the composition rule for rigid-body motions that

$$H((R_1, \mathbf{b}_1) \circ (R_2, \mathbf{b}_2)) = H(R_1, \mathbf{b}_1)H(R_2, \mathbf{b}_2)$$

[2]Conjugation of a motion $g = (R, \mathbf{x})$ by a motion $h = (Q, \mathbf{y})$ is defined as $h \circ g \circ h^{-1}$.

[3]The theory of screws was developed by Sir Robert Stawell Ball (1840–1913) [13].

Likewise, the inverse of a homogeneous transformation matrix represents the inverse of a motion:

$$H((R,\mathbf{b})^{-1}) = [H(R,\mathbf{b})]^{-1}$$

In this notation, vectors in $\mathbb{R}^3$ are augmented by appending a "1" to form a vector

$$\mathbf{X} = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$$

The following are then equivalent expressions:

$$\mathbf{Y} = H(R,\mathbf{b})\mathbf{X} \leftrightarrow \mathbf{y} = R\mathbf{x} + \mathbf{b}$$

Homogeneous transforms representing pure translations and rotations are respectively written in the form

$$\text{trans}(\mathbf{n}, d) = \begin{pmatrix} \mathbb{I} & d\mathbf{n} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

and

$$\text{rot}(\mathbf{n}, \mathbf{p}, \theta) = \begin{pmatrix} e^{\theta N} & (\mathbb{I} - e^{\theta N})\mathbf{p} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Rotations around, and translations along, the same axis commute, and the homogeneous transform for a general rigid-body motion along screw axis $(\mathbf{n}, \mathbf{p})$ is given as

$$\text{rot}(\mathbf{n}, \mathbf{p}, \theta)\text{trans}(\mathbf{n}, d) = \text{trans}(\mathbf{n}, d)\text{rot}(\mathbf{n}, \mathbf{p}, \theta) = \begin{pmatrix} e^{\theta N} & (\mathbb{I} - e^{\theta N})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{2.19}$$

### 2.3.2 The Denavit-Hartenberg Parameters in Robotics

The Denavit-Hartenberg (D-H) framework is a method for assigning frames of reference to a serial robot arm constructed of sequential rotary (and/or translational) joints connected with rigid links [15]. If the robot arm is imagined at any fixed time, the axes about which the joints turn are viewed as lines in space. In the most general case, these lines will be skew, and in degenerate cases, they can be parallel or intersect.

In the D-H framework, a frame of reference is assigned to each link of the robot at the joint where it meets the previous link. The $z$-axis of the $i$th D-H frame points along the $i$th joint axis. Since a robot arm is usually attached to a base, there is no ambiguity in terms of which of the two ($\pm$) directions along the joint axis should be chosen, i.e., the "up" direction for the first joint is chosen. Since the $(i+1)$st joint axis in space will generally be skew relative to axis $i$, a unique $x$-axis is assigned to frame $i$, by defining it to be the unit vector pointing in the direction of the shortest line segment from axis $i$ to axis $i+1$. This segment intersects both axes orthogonally. In addition to completely defining the relative orientation of the $i$th frame relative to the $(i-1)$st, it also provides the relative position of the origin of this frame.

The D-H parameters, which completely specify this model, are [1]:

- The distance from joint axis $i$ to axis $i+1$ as measured along their mutual normal. This distance is denoted as $a_i$.
- The angle between the projection of joint axes $i$ and $i+1$ in the plane of their common normal. The sense of this angle is measured counterclockwise around their mutual normal originating at axis $i$ and terminating at axis $i+1$. This angle is denoted as $\alpha_i$.
- The distance between where the common normal of joint axes $i-1$ and $i$, and that of joint axes $i$ and $i+1$ intersect joint axis $i$, as measured along joint axis $i$. This is denoted as $d_i$.

- The angle between the common normal of joint axes $i - 1$ and $i$, and the common normal of joint axes $i$ and $i + 1$. This is denoted as $\theta_i$, and has positive sense when rotation about axis $i$ is counterclockwise.

Hence, given all the parameters $\{a_i, \alpha_i, d_i, \theta_i\}$ for all the links in the robot, together with how the base of the robot is situated in space, one can completely specify the geometry of the arm at any fixed time. Generally, $\theta_i$ is the only parameter that depends on time.

In order to solve the *forward kinematics* problem, which is to find the position and orientation of the distal end of the arm relative to the base, the homogeneous transformations of the relative displacements from one D-H frame to another are multiplied sequentially. This is written as

$$H_N^0 = H_1^0 H_2^1 \cdots H_N^{N-1}$$

The relative transformation, $H_i^{i-1}$ from frame $i - 1$ to frame $i$ is performed by first rotating about the $x$-axis of frame $i - 1$ by $\alpha_{i-1}$, then translating along this same axis by $a_{i-1}$. Next we rotate about the $z$-axis of frame $i$ by $\theta_i$ and translate along the same axis by $d_i$. Since all these trasformations are relative, they are multiplied sequentially on the right as rotations (and translations) about (and along) natural basis vectors. Furthermore, since the rotations and translations appear as two screw motions (translations and rotations along the same axis), we write

$$H_i^{i-1} = \text{Screw}(\mathbf{e}_1, a_{i-1}, \alpha_{i-1})\text{Screw}(\mathbf{e}_3, d_i, \theta_i)$$

where in this context

$$\text{Screw}(\mathbf{v}, c, \gamma) = \text{rot}(\mathbf{v}, \mathbf{0}, \gamma)\text{trans}(\mathbf{v}, c)$$

Explicitly,

$$H_i^{i-1}(a_{i-1}, \alpha_{i-1}, d_i, \theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i \cos\alpha_{i-1} & \cos\theta_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i \sin\alpha_{i-1} \\ \sin\theta_i \sin\alpha_{i-1} & \cos\theta_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i \cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

## 2.4 Infinitesimal Motions and Associated Jacobian Matrices

A "small motion" is one which describes the relative displacement of a rigid body at two times differing by only an instant. Small rigid-body motions, whether pure rotations or combinations of rotation and translation, differ from large displacements in both their properties and the way they are described. This section explores these small motions in detail.

### 2.4.1 Angular Velocity and Jacobians Associated with Parametrized Rotations

It is clear from Euler's theorem that when $|\theta| \ll 0$, a rotation matrix reduces to the form

$$A_E(\theta, \mathbf{n}) = \mathbb{I} + \theta N$$

This means that for small rotation angles $\theta_1$ and $\theta_2$, rotations commute:

$$A_E(\theta_1, \mathbf{n}_1)A_E(\theta_2, \mathbf{n}_2) = \mathbb{I} + \theta_1 N_1 + \theta_2 N_2 = A_E(\theta_2, \mathbf{n}_2)A_E(\theta_1, \mathbf{n}_1)$$

Given two frames of reference, one of which is fixed in space and the other of which is rotating relative to it, a rotation matrix describing the orientation of the rotating frame as seen in the fixed frame is written as $R(t)$ at each time $t$. One connects the concepts of small rotations and angular velocity by observing that if $\mathbf{x}_0$ is a fixed (constant) position vector in the rotating frame of reference, then the position of the same point as seen in a frame of reference fixed in space with the same origin as the rotating frame is related to this as

$$\mathbf{x} = R(t)\mathbf{x}_0 \leftrightarrow \mathbf{x}_0 = R^T(t)\mathbf{x}$$

The velocity as seen in the frame fixed in space is then

$$\mathbf{v} = \dot{\mathbf{x}} = \dot{R}\mathbf{x}_0 = \dot{R}R^T\mathbf{x} \tag{2.20}$$

Observing that since $R$ is a rotation matrix,

$$\frac{d}{dt}(RR^T) = \frac{d}{dt}(\mathbb{I}) = 0$$

and one writes

$$\dot{R}R^T = -R\dot{R}^T = -(\dot{R}R^T)^T$$

Due to the skew-symmetry of this matrix, we can rewrite Equation (2.20) in the form most familiar to engineers:

$$\mathbf{v} = \boldsymbol{\omega}_L \times \mathbf{x}$$

and the notation $\boldsymbol{\omega}_L = \text{vect}(\dot{R}R^T)$ is used. The vector $\boldsymbol{\omega}_L$ is the angular velocity as seen in the space-fixed frame of reference (i.e., the frame in which the moving frame appears to have orientation given by $R$). The subscript $L$ is not conventional but serves as a reminder that the $\dot{R}$ appears on the "left" of the $R^T$ inside the vect$(\cdot)$. In contrast, the angular velocity as seen in the rotating frame of reference (where the orientation of the moving frame appears to have orientation given by the identity rotation) is the dual vector of $R^T\dot{R}$, which is also a skew-symmetric matrix. This is denoted as $\boldsymbol{\omega}_R$ (where the subscript $R$ stands for "right"). Therefore we have

$$\boldsymbol{\omega}_R = \text{vect}(R^T\dot{R}) = \text{vect}(R^T(\dot{R}R^T)R) = R^T\boldsymbol{\omega}_L$$

This is because for any skew-symmetric matrix $X$ and any rotation matrix $R$, $\text{vect}(R^TXR) = R^T\text{vect}(X)$. Another way to write this is

$$\boldsymbol{\omega}_L = R\boldsymbol{\omega}_R$$

In other words, the angular velocity as seen in the frame of reference fixed in space is obtained from the angular velocity as seen in the rotating frame in the same way in which the absolute position is obtained from the relative position.

When a time-varying rotation matrix is parametrized as

$$R(t) = A(q_1(t), q_2(t), q_3(t)) = A(\mathbf{q}(t))$$

then by the chain rule from calculus, one has

$$\dot{R} = \frac{\partial A}{\partial q_1}\dot{q}_1 + \frac{\partial A}{\partial q_2}\dot{q}_2 + \frac{\partial A}{\partial q_3}\dot{q}_3$$

Multiplying on the right by $R^T$ and extracting the dual vector from both sides, one finds that

$$\boldsymbol{\omega}_L = J_L(A(\mathbf{q}))\dot{\mathbf{q}} \tag{2.21}$$

where one can define [6]:

$$J_L(A(\mathbf{q})) = \left[\text{vect}\left(\frac{\partial A}{\partial q_1}A^T\right), \text{vect}\left(\frac{\partial A}{\partial q_2}A^T\right), \text{vect}\left(\frac{\partial A}{\partial q_3}A^T\right)\right]$$

Similarly,

$$\boldsymbol{\omega}_R = J_R(A(\mathbf{q}))\dot{\mathbf{q}} \tag{2.22}$$

where

$$J_R(A(\mathbf{q})) = \left[\text{vect}\left(A^T\frac{\partial A}{\partial q_1}\right), \text{vect}\left(A^T\frac{\partial A}{\partial q_2}\right), \text{vect}\left(A^T\frac{\partial A}{\partial q_3}\right)\right]$$

These two Jacobian matrices are related as

$$J_L = AJ_R \tag{2.23}$$

It is easy to verify from the above expressions that for an arbitrary constant rotation $R_0 \in SO(3)$:

$$J_L(R_0A(\mathbf{q})) = R_0J_L(A(\mathbf{q})), \quad J_L(A(\mathbf{q})R_0) = J_L(A(\mathbf{q}))$$
$$J_R(R_0A(\mathbf{q})) = J_R(A(\mathbf{q})), \quad J_R(A(\mathbf{q})R_0) = R_0^T J_R(A(\mathbf{q}))$$

In the following subsection we provide the explicit forms for the left and right Jacobians for the $ZXZ$ Euler-angle parametrization. Analogous computations for the other parametrizations discussed earlier in this chapter can be found in [6].

### 2.4.2 The Jacobians for $ZXZ$ Euler Angles

In this subsection we explicitly calculate the Jacobian matrices $J_L$ and $J_R$ for the $ZXZ$ Euler angles. In this case, $A(\alpha, \beta, \gamma) = R_3(\alpha)R_1(\beta)R_3(\gamma)$, and the skew-symmetric matrices whose dual vectors form the columns of the Jacobian matrix $J_L$ are given as[4]:

$$\begin{aligned}
\frac{\partial A}{\partial \alpha}A^T &= (R_3'(\alpha)R_1(\beta)R_3(\gamma))(R_3(-\gamma)R_1(-\beta)R_3(-\alpha)) = R_3'(\alpha)R_3(-\alpha) \\
\frac{\partial A}{\partial \beta}A^T &= (R_3(\alpha)R_1'(\beta)R_3(\gamma))(R_3(-\gamma)R_1(-\beta)R_3(-\alpha)) \\
&= R_3(\alpha)(R_1'(\beta)R_1(-\beta))R_3(-\alpha) \\
\frac{\partial A}{\partial \gamma}A^T &= R_3(\alpha)R_1(\beta)(R_3'(\gamma)(R_3(-\gamma))R_1(-\beta)R_3(-\alpha)
\end{aligned}$$

Noting that $\text{vect}(R_i'R_i^T) = \mathbf{e}_i$ regardless of the value of the parameter, and using the rule $\text{vect}(RXR^T) = R\text{vect}(X)$, one finds

$$J_L(\alpha, \beta, \gamma) = [\mathbf{e}_3, R_3(\alpha)\mathbf{e}_1, R_3(\alpha)R_1(\beta)\mathbf{e}_3]$$

This is written explicitly as

$$J_L(\alpha, \beta, \gamma) = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha\sin\beta \\ 0 & \sin\alpha & -\cos\alpha\sin\beta \\ 1 & 0 & \cos\beta \end{pmatrix}$$

The Jacobian $J_R$ can be derived similarly, or one can calculate it easily from $J_L$ as [6]

$$J_R = A^TJ_L = [R_3(-\gamma)R_1(-\beta)\mathbf{e}_3, R_3(-\gamma)\mathbf{e}_1, \mathbf{e}_3]$$

[4]For one-parameter rotations we use the notation $'$ to denote differentiation with respect to the parameter.

Explicitly, this is

$$J_R = \begin{pmatrix} \sin\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta\cos\gamma & -\sin\gamma & 0 \\ \cos\beta & 0 & 1 \end{pmatrix}$$

These Jacobian matrices are important because they relate rates of change of Euler angles to angular velocities. When the determinants of these matrices become singular, the Euler-angle description of rotation becomes degenerate.

### 2.4.3 Infinitesimal Rigid-Body Motions

As with pure rotations, the matrix exponential can be used to describe general rigid-body motions [2,14]. For "small" motions the matrix exponential description is approximated well when truncated at the first two terms:

$$\exp\left[\begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{0}^T & 0 \end{pmatrix} \Delta t\right] \approx \mathbb{I} + \begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{0}^T & 0 \end{pmatrix} \Delta t \tag{2.24}$$

Here $\Omega = -\Omega^T$ and $\text{vect}(\Omega) = \boldsymbol{\omega}$ describe the rotational part of the displacement. Since the second term in Equation (2.24) consists mostly of zeros, it is common to extract the information necessary to describe the motion as

$$\begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{0}^T & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{v} \end{pmatrix}$$

This six-dimensional vector is called an *infinitessimal* screw motion or *infinitessimal twist*.

Given a homogeneous transform

$$H(\mathbf{q}) = \begin{pmatrix} R(\mathbf{q}) & \mathbf{b}(\mathbf{q}) \\ \mathbf{0}^T & 0 \end{pmatrix}$$

parametrized with $(q_1, \ldots, q_6)$, which we write as a vector $\mathbf{q} \in I\!R^6$, one can express the homogeneous transform corresponding to a slightly changed set of parameters as the truncated Taylor series

$$H(\mathbf{q} + \Delta\mathbf{q}) = H(\mathbf{q}) + \sum_{i=1}^{6} \Delta q_i \frac{\partial H}{\partial q_i}(\mathbf{q})$$

This result can be shifted to the identity transformation by multiplying on the right or left by $H^{-1}$ to define an equivalent relative infinitessimal motion. In this case we write [6]

$$\begin{pmatrix} \boldsymbol{\omega}_L \\ \mathbf{v}_L \end{pmatrix} = \mathcal{J}_L(\mathbf{q})\dot{\mathbf{q}} \quad \text{where} \quad \mathcal{J}_L(\mathbf{q}) = \left[\left(\frac{\partial H}{\partial q_1} H^{-1}\right)^{\vee}, \ldots, \left(\frac{\partial H}{\partial q_6} H^{-1}\right)^{\vee}\right] \tag{2.25}$$

where $\boldsymbol{\omega}_L$ is defined as before in the case of pure rotation and

$$\mathbf{v}_L = -\boldsymbol{\omega}_L \times \mathbf{b} + \dot{\mathbf{b}} \tag{2.26}$$

Similarly,

$$\begin{pmatrix} \boldsymbol{\omega}_R \\ \mathbf{v}_R \end{pmatrix} = \mathcal{J}_R(\mathbf{q})\dot{\mathbf{q}} \quad \text{where} \quad \mathcal{J}_R(\mathbf{q}) = \left[\left(H^{-1}\frac{\partial H}{\partial q_1}\right)^{\vee}, \ldots, \left(H^{-1}\frac{\partial H}{\partial q_6}\right)^{\vee}\right] \tag{2.27}$$

Here

$$\mathbf{v}_R = R^T \dot{\mathbf{b}}$$

The left and right Jacobian matrices are related as

$$\mathcal{J}_L(H) = [Ad(H)]\mathcal{J}_R(H) \tag{2.28}$$

where the matrix $[Ad(H)]$ is called the *adjoint* and is written as [2, 6, 8]:

$$[Ad(H)] = \begin{pmatrix} R & 0 \\ BR & R \end{pmatrix} \tag{2.29}$$

The matrix $B$ is skew-symmetric, and $\text{vect}(B) = \mathbf{b}$.

Note that when the rotations are parametrized as $R = R(q_1, q_2, q_3)$ and the translations are parametrized using Cartesian coordinates $\mathbf{b}(q_4, q_5, q_6) = [q_4, q_5, q_6]^T$, one finds that

$$\mathcal{J}_R = \begin{pmatrix} J_R & 0 \\ 0 & R^T \end{pmatrix} \quad \text{and} \quad \mathcal{J}_L = \begin{pmatrix} J_L & 0 \\ BJ_L & \mathbb{I} \end{pmatrix} \tag{2.30}$$

where $J_L$ and $J_R$ are the left and right Jacobians for the case of rotation.

Given

$$H_0 = \begin{pmatrix} R_0 & \mathbf{b}_0 \\ \mathbf{0}^T & 1 \end{pmatrix}$$

$$\mathcal{J}_L(HH_0) = \left[ \left( \frac{\partial H}{\partial q_1} H_0 (HH_0)^{-1} \right)^{\vee} \cdots \left( \frac{\partial H}{\partial q_6} H_0 (HH_0)^{-1} \right)^{\vee} \right]$$

Since $(HH_0)^{-1} = H_0^{-1} H^{-1}$, and $H_0 H_0^{-1} = 1$, we have that $\mathcal{J}_L(HH_0) = \mathcal{J}_L(H)$.

Similarly,

$$\mathcal{J}_L(H_0 H) = \left[ \left( H_0 \frac{\partial H}{\partial q_1} H^{-1} H_0^{-1} \right)^{\vee} \cdots \left( H_0 \frac{\partial H}{\partial q_6} H^{-1} H_0^{-1} \right)^{\vee} \right]$$

where

$$\left( H_0 \frac{\partial H}{\partial q_i} H^{-1} H_0^{-1} \right)^{\vee} = [Ad(H_0)] \left( \frac{\partial H}{\partial q_i} H^{-1} \right)^{\vee}$$

Therefore,

$$\mathcal{J}_L(H_0 H) = [Ad(H_0)]\mathcal{J}_L(H).$$

Analogous expressions can be written for $\mathcal{J}_R$, which is left invariant.

## Further Reading

This chapter has provided a brief overview of rigid-body kinematics. A number of excellent textbooks on kinematics and robotics including [1–4,8,16–18] treat this material in greater depth. Other classic works from a number of fields in which rotations are described include [5,7,9–11]. The interested reader is encouraged to review these materials.

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